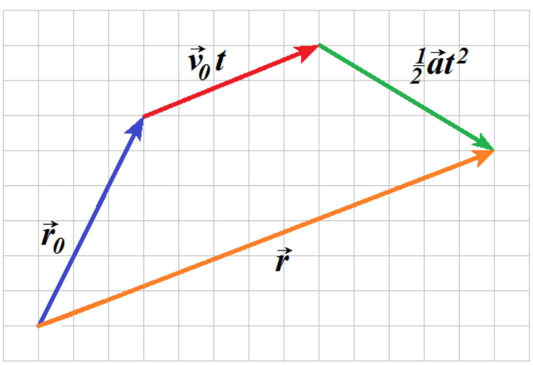
Part 4: Two-Dimensional Kinematics

University Physics V1 (Openstax): Chapter 4
Physics for Engineers & Scientists (Giancoli): Chapter 3

Two-Dimensional Quantities *(Everything but time (t) is now a vector).*

Quantity	One Dimension	Two Dimensions
Position	x (or y)	$\vec{r} = x\hat{\imath} + y\hat{\jmath}$
Initial Position	x_0 (or y_0)	$\vec{r}_0 = x_0\hat{\imath} + y_0\hat{\jmath}$
Displacement	Δx (or Δy)	$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$
Average Velocity	$v_{avg} = \frac{\Delta x}{\Delta t}$	$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} = v_{x-avg}\hat{\imath} + v_{y-avg}\hat{\jmath}$
Average Acceleration	$a_{avg} = \frac{\Delta v}{\Delta t}$	$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\imath} + \frac{\Delta v_y}{\Delta t}\hat{\jmath} = a_{x-avg}\hat{\imath} + a_{y-avg}\hat{\jmath}$
Const a equation #1 (no x)	$v = v_0 + at$	$\vec{v} = \vec{v}_0 + \vec{a}t$
Const a equation #2 (no a)	$x = x_0 + \frac{1}{2}(v + v_0)t$	$\vec{r} = \vec{r}_0 + \frac{1}{2}(\vec{v} + \vec{v}_0)t$
Const a equation #3 (no v)	$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$
Const a equation #4 (no t)	$v^2 = v_0^2 + 2a(x - x_0)$	$\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}\cdot(\vec{r} - \vec{r}_0)$

Solving Problems



- If the vectors are all in the same direction, then you can treat it 1-dimensionally.
- If not, then you split everything into components and solve x and y separately, then recombine back into vectors at the end.
- The x and y equations are linked by time and/or any angles given. Apart from that, the x- and y-components are independent from each other.
- Projectile motion problems (i.e. falling near the Earth's surface): $a_{horizontal} = 0$, $a_{vertical} = -9.80\ m/s^2$ (downward)

Example: A player kicks a ball at rest. The ball remains in contact with the kicker's foot for 0.0500s, during which time it experiences an acceleration of 340.0m/s². The ball is launched at an angle of 51.0° above the ground. Determine the horizontal and vertical components of the launch velocity.

$$\vec{v}_0 = 0 \quad t = 0.0500\ s \quad \vec{a} = 340.0\angle 51.0° \quad \vec{v} = ?$$

$$\vec{v} = \vec{v}_0 + \vec{a}t = \vec{a}t = \left(340.0\frac{m}{s^2}\right)(0.0500\ s) = 17.0\frac{m}{s}$$

$$v_x = v\cdot \text{Cos}(\theta) = \left(17.0\frac{m}{s}\right)\text{Cos}(51.0°) = 10.698\frac{m}{s} \Rightarrow 10.7\frac{m}{s}$$

$$v_y = v\cdot \text{Sin}(\theta) = \left(17.0\frac{m}{s}\right)\text{Sin}(51.0°) = 13.211\frac{m}{s} \Rightarrow 13.2\frac{m}{s}$$

Example: For the previous example, after the ball leaves the kicker's foot, how far from the initial position will it land?

x-components: $x_0 = 0$ $v_{0x} = 10.698$ m/s $a_x = 0$ $v_x = v_{0x}$

As a_x=0, the only equation available is: $x = x_0 + v_x t = v_x t$

This would give us the answer ...if we had t $\rightarrow$ Need t from y-components.

y-components: What is v_y when it lands?

When $y = y_0$, then $v = -v_0$. $\{ v_y^2 = v_{oy}^2 + 2a_y \underbrace{(y - y_0)}_{0} \rightarrow v_y^2 = v_{0y}^2 \}$

$y_0 = y = 0$ $v_{oy} = 13.211$ m/s $v_y = - v_{oy} = -13.211$ m/s $a_y = -g = -9.80$ m/s^2 t = ???

$$v_y = v_{0y} + a_y t \qquad v_y - v_{0y} = a_y t$$

$$t = \frac{v_y - v_{0y}}{a_y} = \frac{v_y - v_{0y}}{-g} = \frac{-v_{0y} - v_{0y}}{-g} = \frac{-2v_{0y}}{-g} = \frac{2v_{0y}}{g} = \frac{2\left(13.211\,\frac{m}{s}\right)}{9.80\ m/s^2} = 2.6961\ s$$

$$x = v_{0x} t = \left(10.698\ \frac{m}{s}\right)(2.6961\ s) = 28.843\ m \Rightarrow 28.8\ m.$$

Example: For the previous examples, after the ball leaves the kicker's foot, how high will the ball go? (i.e. determine the maximum height)

What do we know about maximum displacement? $v_y = 0$

$y_0 = 0$ $v_{oy} = 13.211$ m/s $v_y = 0$ $a_y = -g = -9.80$ m/s^2 y = ??? (no t)

$$v_y^2 = v_{oy}^2 + 2a_y(y - y_0) \qquad 0 = v_{oy}^2 + 2a_y y \qquad -v_{oy}^2 = 2a_y y$$

$$y = \frac{-v_{oy}^2}{2a_y} = \frac{-v_{oy}^2}{-2g} = \frac{v_{oy}^2}{2g} = \frac{\left(13.211\,\frac{m}{s}\right)^2}{2\left(9.80\,\frac{m}{s^2}\right)} = 8.90462\ m \Rightarrow 8.90\ m$$

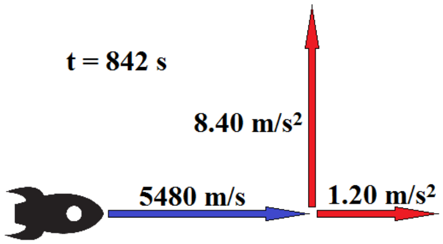
Note: we could have found t first.

$$v_y = v_{0y} + a_y t \qquad v_y - v_{0y} = a_y t$$

$$t = \frac{v_y - v_{0y}}{a_y} = \frac{-v_{0y}}{-g} = \frac{v_{0y}}{g} = \frac{13.211\frac{m}{s}}{9.80\frac{m}{s^2}} = 1.3481\ s \quad \text{(half the time of the previous example)}$$

$$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 = 0 + \left(13.211\,\frac{m}{s}\right)(1.3481\ s) + \frac{1}{2}\left(-9.80\,\frac{m}{s^2}\right)(1.3481\ s)^2 = 8.90462\ m$$

Example: A spacecraft is travelling with a velocity of v_{0x} = 5480 m/s along the +x direction. Two engines are turned on for a time of 842 s. One engine gives the spacecraft an acceleration in the +x direction of a_x = 1.20 m/s², while the other gives it an acceleration in the +y direction of 8.40 m/s². At the end of the firing, find (a) v_x and (b) v_y.



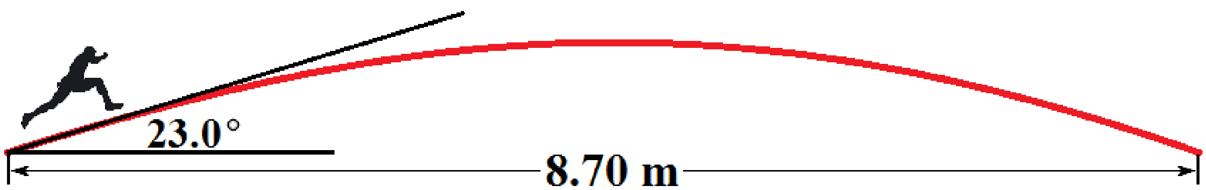
t = 842s v_{ox} = 5480m/s v_{oy} = 0 a_x = 1.20m/s² a_y = 8.40m/s²

$$v_x = v_{0x} + a_x t = \left(5480\frac{m}{s}\right) + \left(1.20\frac{m}{s^2}\right)(842\ s) = 6490\frac{m}{s}$$

$$v_y = v_{0y} + a_y t = \left(0\frac{m}{s}\right) + \left(8.40\frac{m}{s^2}\right)(842\ s) = 7070\frac{m}{s}$$

If the quadratic equation gives you trouble, you can find your answer in another way!

Example: An Olympic jumper leaves the ground at an angle of 23.0° and travels through the air for a horizontal distance of 8.70 m before landing. What is the takeoff speed of the jumper?



$x_o = 0$ x = 8.70m a_x=0 $v_{ox} = v_x = v_o\text{Cos}(23.0°)$

$y_o = 0$ y = 0 $v_{oy} = v_o\text{Sin}(23.0°)$ $v_y = -v_{oy} = -v_o\text{Sin}(23.0°)$ $a_y=-g$

The x and y equations are linked via t (so we need equations with t), but we have 2 unknowns (v_0 and t). Need 2 equations (x-comp and y-comp) that have both v_0 and t.

x-components: $x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 = v_{0x}t = v_0\text{Cos}(23.0°)t$ $\quad t = \frac{x}{v_0\text{Cos}(23.0°)}$

y-components: $v_y = v_{0y} + a_y t$ $\quad -v_0\text{Sin}(23.0°) = v_0\text{Sin}(23.0°) - gt$

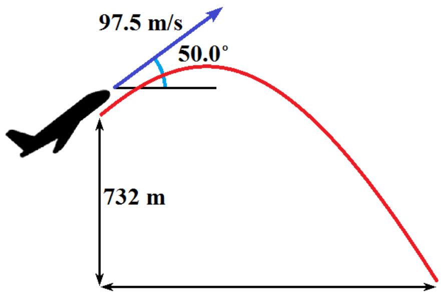
$$-2v_0\text{Sin}(23.0°) = -gt \qquad 2v_0\text{Sin}(23.0°) = gt$$

Plug in t (from x-comp): $2v_0\text{Sin}(23.0°) = g\left(\frac{x}{v_0\text{Cos}(23.0°)}\right)$ $\quad 2v_0\text{Sin}(23.0°) = \frac{gx}{v_0\text{Cos}(23.0°)}$

$$v_0 = \frac{gx}{v_0 2\text{Sin}(23.0°)\text{Cos}(23.0°)} \qquad v_0^2 = \frac{gx}{2\text{Sin}(23.0°)\text{Cos}(23.0°)}$$

$$v_0 = \sqrt{\frac{gx}{2\text{Sin}(23.0°)\text{Cos}(23.0°)}} = \sqrt{\frac{\left(9.80\frac{m}{s^2}\right)(8.70\ m)}{2\text{Sin}(23.0°)\text{Cos}(23.0°)}} = 10.887\frac{m}{s} \Rightarrow 10.9\frac{m}{s}$$

Example: An airplane with a speed of 97.5m/s is climbing upwards at an angle of 50.0° with respect to the horizontal. When the plane's altitude is 732m the pilot releases a package. (a) Calculate the distance along the ground measured from a point directly beneath the point of release, to where the package hits the earth. (b) Relative to the ground, determine the angle of the velocity vector just before impact.



$x_0 = 0$ $x = ???$ $a_x = 0$

$v_{0x} = (97.5\text{ m/s})\text{Cos}(50.0°) = 62.6718\text{ m/s}$

$v_x = v_{0x} = 62.6718\text{ m/s}$

$y_0 = 732\text{ m}$ $y = 0$

$v_{0y} = (97.5\text{ m/s})\text{Sin}(50.0°) = 74.6893\text{ m/s}$

$v_y =$ $a_y = -g$ $t =$

x-components (find x): $x = x_0 + v_x t = v_x t$ *{Need t.}*

y-components (find t, no v): $y = y_0 + v_{oy}t + \frac{1}{2}a_y t^2$ *{Solve with quadratic equation}*

$$0 = 732\text{ m} + \left(74.6893\frac{\text{m}}{\text{s}}\right)t - \frac{1}{2}\left(9.80\frac{\text{m}}{\text{s}^2}\right)t^2 \qquad 0 = 732\text{ m} + \left(74.6893\frac{\text{m}}{\text{s}}\right)t - \left(4.90\frac{\text{m}}{\text{s}^2}\right)t^2$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-74.6893\frac{\text{m}}{\text{s}} \pm \sqrt{\left(74.6893\frac{\text{m}}{\text{s}}\right)^2 - 4\left(-4.90\frac{\text{m}}{\text{s}^2}\right)(732\text{ m})}}{2\left(-4.90\frac{\text{m}}{\text{s}^2}\right)}$$

$$t = \frac{-74.6893\frac{\text{m}}{\text{s}} \pm \sqrt{5578.4915\frac{\text{m}^2}{\text{s}^2} + 14347.2\frac{\text{m}^2}{\text{s}^2}}}{-9.80\frac{\text{m}}{\text{s}^2}} \qquad t = \frac{-74.6893\frac{\text{m}}{\text{s}} \pm \sqrt{19925.6915\frac{\text{m}^2}{\text{s}^2}}}{-9.80\frac{\text{m}}{\text{s}^2}}$$

$$t = \frac{-74.6893\frac{\text{m}}{\text{s}} \pm 141.1584\frac{\text{m}}{\text{s}}}{-9.80\frac{\text{m}}{\text{s}^2}} \qquad t_1 = \frac{-74.6893\frac{\text{m}}{\text{s}} + 141.1584\frac{\text{m}}{\text{s}}}{-9.80\frac{\text{m}}{\text{s}^2}} = \frac{66.4691\frac{\text{m}}{\text{s}}}{-9.80\frac{\text{m}}{\text{s}^2}} = -6.78256\text{ s}$$

$$t_2 = \frac{-74.6893\frac{\text{m}}{\text{s}} - 141.1584\frac{\text{m}}{\text{s}}}{-9.80\frac{\text{m}}{\text{s}^2}} = \frac{-215.8477\frac{\text{m}}{\text{s}}}{-9.80\frac{\text{m}}{\text{s}^2}} = 22.0253\text{ s}$$

Now plug the value of t into the original equation to get x:

$$x = v_x t = \left(62.6718\frac{\text{m}}{\text{s}}\right)(22.0253\text{ s}) = 1380.36 \Rightarrow 1380\text{ m}$$

The value of t can also be used to get v_y, which is needed to produce the velocities angle (part b).

$$v_y = v_{0y} + a_y t = \left(74.6893\frac{\text{m}}{\text{s}}\right) + \left(-9.80\frac{\text{m}}{\text{s}^2}\right)(22.0253\text{ s}) = -141.159\frac{\text{m}}{\text{s}}$$

$$\theta = \text{Tan}^{-1}\left(\frac{v_y}{v_x}\right) = \text{Tan}^{-1}\left(\frac{-141.159\frac{\text{m}}{\text{s}}}{62.6718\frac{\text{m}}{\text{s}}}\right) = -66.0597° \Rightarrow -66.1°$$

Alternatively (instead of using quadratic equation) find v first, then t.

$$v_y^2 = v_{0y}^2 + 2a_y(y - y_0) = \left(74.6893\frac{\text{m}}{\text{s}}\right)^2 + 2\left(-9.80\frac{m}{s^2}\right)(0 - 732\,m) = 19925.69\frac{m^2}{s^2}$$

$$v_y = -\sqrt{19925.69\frac{m^2}{s^2}} = -141.158\frac{m}{s} \qquad \mathrm{v_y = v_{0y} + a_y t} \qquad \mathrm{v_y - v_{0y} = a_y t = -gt}$$

$$\mathrm{t = \frac{v_y - v_{0y}}{-g} = \frac{-141.158\frac{m}{s} - 74.6893\frac{m}{s}}{-9.80\frac{m}{s^2}} = 22.0252\ s}$$

$$\mathrm{x = v_x t = \left(62.6718\frac{m}{s}\right)(22.0252\ s) = 1380.36 \Rightarrow 1380\ m}$$

$$\theta = \mathrm{Tan^{-1}}\left(\frac{\mathrm{v_y}}{\mathrm{v_x}}\right) = \mathrm{Tan^{-1}}\left(\frac{-141.158\frac{\mathrm{m}}{\mathrm{s}}}{62.6718\frac{\mathrm{m}}{\mathrm{s}}}\right) = -66.0596° \Rightarrow -66.1°$$

If the quadratic equation gives you trouble, you can find your answer in another way!

Exercises

1. A batter is a baseball game hits a pitch. It leaves the bat at a height of 1.00 m with a speed of 31.5 m/s, leaving at an angle of 29.8 degrees above horizontal. Determine the height of the ball as it passes over 2nd base, a distance of 38.4 m away.

2. A car is heading in the positive x-direction at 31.9 m/s when it starts to turn, experiencing an acceleration of $\vec{a} = \left(-1.15\frac{m}{s^2}\right)\hat{\imath} + \left(1.75\frac{m}{s^2}\right)\hat{\jmath}$. How long does it take, starting from when the acceleration begins, until the car has turned 45.0° from its initial course?

3. A ball rolls off of a flat table that is 1.10 m in height. It lands 0.535 m from the edge of the table. Determine the speed of the ball as it rolled off the table.

Exercise Solutions

1. A batter is a baseball game hits a pitch. It leaves the bat at a height of 1.00 m with a speed of 31.5 m/s, leaving at an angle of 29.8 degrees above horizontal. Determine the height of the ball as it passes over 2nd base, a distance of 38.4 m away.

   $x_0 = 0 \quad x = 38.4\,m \quad v_{0x} = 31.5 \cdot Cos(29.8°) = 27.335\frac{m}{s} \quad v_x = 27.335\frac{m}{s} \quad a_x = 0$ t = ?

   In freefall, the horizontal acceleration is zero. Consequently, the x-component of the velocity never changes ($v_x = v_{0x}$). We also know that in freefall, $a_y = -g$.

   $y_0 = 1.00\,m \quad \boldsymbol{y} = ??? \quad v_{0y} = 31.5 \cdot Sin(29.8°) = 15.655\frac{m}{s} \quad v_y = ? \quad a_y = -g = -9.80\frac{m}{s^2}$

   In the y-direction, we are looking for y, but we are missing both v_y and t. This prevents us from solving for y directly. We need one of those two values. We can get t from the x-coordinates, and there is only one equation, which simplifies as $x_0=0$.

   $x = x_0 + v_x t = v_x t \qquad t = \frac{x}{v_x} = \frac{38.4\,m}{27.335\frac{m}{s}} = 1.405\,s$

   With the value of t, we can now solve for y, using the equation that doesn't have v_y in it.

   $y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = (1.00\,m) + \left(15.655\frac{m}{s}\right)(1.405\,s) + \frac{1}{2}\left(-9.80\frac{m}{s^2}\right)(1.405\,s)^2 = 13.3\,m$

2. A car is heading in the positive x-direction at 31.9 m/s when it starts to turn, experiencing an acceleration of $\vec{a} = \left(-1.15\frac{m}{s^2}\right)\hat{\imath} + \left(1.75\frac{m}{s^2}\right)\hat{\jmath}$. How long does it take, starting from when the acceleration begins, until the car has turned 45.0° from its initial course?

   At 45 degrees, the x and y components of the velocity will be the same as $Tan(45°) = \frac{v_y}{v_x} = 1$.

   $1 = \frac{v_y}{v_x} = \frac{v_{0y} + a_y t}{v_{0x} + a_x t} = \frac{a_y t}{v_{0x} + a_x t} \qquad a_y t = v_{0x} + a_x t \qquad a_y t - a_x t = v_{0x} \qquad (a_y - a_x)t = v_{0x}$

   $$t = \frac{v_{0x}}{a_y - a_x} = \frac{\left(31.9\frac{m}{s}\right)}{\left(1.75\frac{m}{s^2}\right) - \left(-1.15\frac{m}{s^2}\right)} = 11.0\,s$$

3. A ball rolls off of a flat table that is 1.10 m in height. It lands 0.535 m from the edge of the table. Determine the speed of the ball as it rolled off the table.

   $x_0 = 0 \quad x = 38.4\,m \quad \boldsymbol{v_{0x} = v_x} = ??? \quad a_x = 0$ t = ?

   In freefall, the horizontal acceleration is zero. Consequently, the x-component of the velocity never changes ($v_x = v_{0x}$). We also know that in freefall, $a_y = -g$. However, to find v_x using the only equation available ($x = x_0 + v_x \cdot t$), we need the value of time when it hits the ground. For this we must use the y-components. Also note that as it is rolling horizontally, $v_{0y} = 0$.

   $y_0 = 1.10\,m \quad y = 0 \quad v_{0y} = 0 \quad v_y = ? \quad a_y = -g = -9.80\frac{m}{s^2}$

   In the y-direction, we are looking for t, and we don't have a value for v_y. So, we use the equation that doesn't have v_y in it. This equation can lead to solving the quadratic equation, but that doesn't happen when v_{0y} is zero

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \qquad 0 = y_0 - \frac{1}{2}gt^2 \qquad y_0 = \frac{1}{2}gt^2 \qquad t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2(1.10\ m)}{9.80\frac{m}{s^2}}} = 0.4738\ s$$

With the value of t, we can now solve for v_{0x}.

$$x = x_0 + v_{0x}t = v_{0x}t \qquad v_{0x} = \frac{x}{t} = \frac{0.535\ \text{m}}{0.4738\ s} = 1.13\frac{m}{s}$$